

SOLITON GAS IN SPACE-CHARGE DOMINATED BEAMS

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Abstract

Based on the Vlasov-Maxwell equations describing the self-consistent nonlinear beam dynamics and collective processes, the evolution of an intense sheet beam propagating through a periodic focusing field has been studied. It has been shown that in the case of a beam with uniform phase space density the Vlasov-Maxwell equations can be replaced exactly by the hydrodynamic equations with a triple adiabatic pressure law coupled to the Maxwell equations. We further demonstrate that starting from the system of hydrodynamic and Maxwell equations a set of coupled nonlinear Schrodinger equations for the slowly varying amplitudes of density waves can be derived. In the case where a parametric resonance between a certain mode of density waves and the external focusing occurs, the slow evolution of the resonant amplitudes in the cold-beam limit is shown to be governed by a system of coupled Gross-Pitaevskii equations. Properties of the nonlinear Schrodinger equation as well as properties of the Gross-Pitaevskii equation are discussed, together with soliton and condensate formation in intense particle beams.

1 Introduction

One of the main goals in the commissioning and operation of modern high energy accelerators and storage rings is the achievement of higher and higher beam currents and charge densities. That is why, of particular importance are the effects of intense self-fields due to space charge and current, influencing the beam propagation, its stability and transport properties. In general, a complete description of collective processes in intense charged particle beams is provided by the Vlasov-Maxwell equations for the self-consistent evolution of the beam distribution function and the electromagnetic fields. Although the analytical basis (as mentioned above) for modelling the dynamics and behaviour of space-charge dominated beams is well established, a thorough and satisfactory understanding of collective processes, detailed equilibrium and formation of patterns and coherent structures is far from being complete.

The present paper may be regarded as the first (to our knowledge) attempt to take a view at the description of the evolution and the collective behaviour of intense charged particle beams from an entirely different perspective, as compared to the ones available in the literature. We will be mainly interested in describing the slow evolution of some coarse-grained quantities that are easily measurable, such as the amplitudes of density waves. Due to the nonlinear wave interaction contingent on the nonlinear coupling between the Vlasov and Maxwell equations, one can expect a formation of nontrivial coherent structure that might be fairly stable in space and time. Here, we show that solitary wave patterns in the beam density distribution are an irrevocable feature, characteristic of intense beams. Moreover, density condensates in the special case where a parametric resonance in terms of wave frequency between a particular mode of the fundamental density waves and the external focusing occurs, can be formed.

The paper is organized as follows. It was previously shown [1] that in the case of a sheet beam with constant phase-space density the Vlasov-Maxwell equations are fully equivalent to a hydrodynamic model with zero heat flow and triple-adiabatic equation-of-state. For consistency, in section 2, we repeat the derivation from a slightly different perspective and then use the hydrodynamic equations as a fundament for the subsequent analysis. In section 3, we consider the case of a smooth focusing where the time variation of the β -function (respectively, the time variation of the density envelope function) can be neglected. Further, we demonstrate that starting from the hydrodynamic equations, and using the renormalization group (RG) technique [2, 3, 4, 5] a system of coupled nonlinear Schrodinger equations for the slowly varying amplitudes of density waves can be derived. The purpose of section 4 is twofold. First, we study the case where a parametric resonance between a particular mode of density waves and a resonant Fourier harmonic of the external focusing (a resonant harmonic from the $\sqrt{\beta}$ -function Fourier decomposition) occurs. It is shown that under certain conditions the evolution of the resonant amplitudes of the forward and backward density waves is governed by a system of coupled Gross-Pitaevskii equations (see e.g. the review [6] and the references therein). Secondly, it is demonstrated that in the non-resonant case the results obtained in the smooth focusing approximation coincide with the ones for a periodic focusing up to second order in the formal perturbation parameter. In section 5, the reduction

of the infinite system of coupled nonlinear Schrodinger equations to a system of two coupled nonlinear Schrodinger equations is discussed. Such reduction is possible if one neglects the effect of all other modes and takes into account only the self-interaction of a single mode with a particular wave number. In the non-resonant case it is shown that the renormalized solution for the beam density describes the process of formation of *holes (cavitons)* in intense particle beams. The case where a parametric resonance is present is more interesting. In the cold-beam limit it is demonstrated that the evolution of the forward and the backward resonant wave amplitudes can be well described by a system of two coupled Gross-Pitaevskii equations. Finally, conclusions are drawn in section 6.

2 Derivation of the Hydrodynamic Model

We begin with the Hamiltonian describing the one-dimensional betatron motion in the presence of space-charge field

$$\widehat{H} = \frac{R}{2}\widehat{p}^2 + \frac{G(\theta)\widehat{x}^2}{2R} + V(\widehat{x};\theta) + \frac{eR}{E_s\beta_s^2}\varphi_{sc}(\widehat{x};\theta), \quad (2.1)$$

where e is the electron charge, R is the mean radius of the accelerator, $G(\theta)$ is the focusing strength of the linear machine lattice, $V(\widehat{x};\theta)$ is a contribution coming from nonlinear lattice elements (sextupoles, octupoles, etc.) and E_s and β_s are the energy and the relative velocity of the synchronous particle respectively. In addition, $\varphi_{sc}(\widehat{x};\theta)$ is the self-field potential due to space-charge, which satisfies the one-dimensional Poisson equation

$$\frac{\partial^2 \varphi_{sc}}{\partial \widehat{x}^2} = -\frac{en}{\epsilon_0} \int d\widehat{p} f(\widehat{x}, \widehat{p}; \theta). \quad (2.2)$$

Here $n = N_p/V_t$ is the density of beam particles (N_p is the number of particles in the beam, while V_t is the volume of the area occupied by the beam in the transverse direction), ϵ_0 is the dielectric susceptibility and $f(\widehat{x}, \widehat{p}; \theta)$ is the distribution function in phase space. Equations (2.1) and (2.2) can be written in an alternate form as

$$\widehat{H} = \frac{R}{2}\widehat{p}^2 + \frac{G(\theta)\widehat{x}^2}{2R} + V(\widehat{x};\theta) + \lambda\widetilde{\varphi}(\widehat{x};\theta), \quad (2.3)$$

$$\frac{\partial^2 \widetilde{\varphi}}{\partial \widehat{x}^2} = - \int d\widehat{p} f(\widehat{x}, \widehat{p}; \theta), \quad (2.4)$$

where

$$\lambda = \frac{Re^2n}{\epsilon_0 E_s \beta_s^2}, \quad (2.5)$$

is the beam perveance, and $\varphi_{sc} = en\widetilde{\varphi}/\epsilon_0$. Note that the beam perveance λ is a dimensionless quantity.

Next, we perform a canonical transformation

$$\widehat{x} = x\sqrt{\beta}, \quad \widehat{p} = \frac{1}{\sqrt{\beta}}(p - \alpha x), \quad (2.6)$$

defined by the generating function

$$F_2(\hat{x}, p; \theta) = \frac{\hat{x}p}{\sqrt{\beta}} - \frac{\alpha\hat{x}^2}{2\beta}, \quad (2.7)$$

where α and β (and γ) are the well-known Twiss parameters satisfying the equations

$$\frac{d\alpha}{d\theta} = \frac{G\beta}{R} - R\gamma, \quad \frac{d\beta}{d\theta} = -2R\alpha, \quad \beta\gamma - \alpha^2 = 1. \quad (2.8)$$

As a result, we obtain the new Hamiltonian

$$H = \frac{\dot{\chi}}{2}(p^2 + x^2) + V(x; \theta) + \lambda\sqrt{\beta}U(x; \theta), \quad (2.9)$$

where the self-field potential $U(x; \theta)$ ($\tilde{\varphi} = \sqrt{\beta}U$) satisfies the equation

$$\frac{\partial^2 U}{\partial x^2} = - \int dp f(x, p; \theta), \quad (2.10)$$

and

$$\dot{\chi} = \frac{d\chi}{d\theta} = \frac{R}{\beta}, \quad (2.11)$$

is the derivative of the phase advance with respect to θ .

We are now ready to write the Vlasov equation for the one-particle distribution function $f(x, p; \theta)$ in the two-dimensional phase space (x, p) . It reads as

$$\frac{\partial f}{\partial \theta} + \dot{\chi}p \frac{\partial f}{\partial x} - \left(\dot{\chi}x + \frac{\partial V}{\partial x} + \lambda\sqrt{\beta} \frac{\partial U}{\partial x} \right) \frac{\partial f}{\partial p} = 0, \quad (2.12)$$

and should be solved self-consistently with the Poisson equation (2.10). Following Davidson et al. [1], we consider the case where the distribution function $f(x, p; \theta)$ is constant (independent of x , p and θ) inside a region in phase space confined by the simply connected boundary curves $p_{(+)}(x; \theta)$ and $p_{(-)}(x; \theta)$, and zero outside. In other words,

$$f(x, p; \theta) = \mathcal{C}, \quad \text{for} \quad p_{(-)}(x; \theta) < p < p_{(+)}(x; \theta), \quad (2.13)$$

and $f(x, p; \theta) = 0$ otherwise. The proof that such solution to the Vlasov equation (2.12) exists can be found in appendix A.

The equations for the boundary curves $p_{(\pm)}(x; \theta)$ can be derived as follows. We substitute the explicit form

$$f(x, p; \theta) = \mathcal{C} \left[\mathcal{H}(p - p_{(-)}(x; \theta)) - \mathcal{H}(p - p_{(+)}(x; \theta)) \right] \quad (2.14)$$

of the distribution function into the Vlasov equation (2.12). Here $\mathcal{H}(z)$ is the well-known Heaviside function. Using the fact that the derivative of the Heaviside function with respect to its argument equals the Dirac δ -function, we obtain

$$- \delta(p - p_{(-)}) \frac{\partial p_{(-)}}{\partial \theta} + \delta(p - p_{(+)}) \frac{\partial p_{(+)}}{\partial \theta} + \dot{\chi}p \left[-\delta(p - p_{(-)}) \frac{\partial p_{(-)}}{\partial x} + \delta(p - p_{(+)}) \frac{\partial p_{(+)}}{\partial x} \right]$$

$$-\left(\dot{\chi}x + \frac{\partial V}{\partial x} + \lambda\sqrt{\beta}\frac{\partial U}{\partial x}\right)\left[\delta(p - p_{(-)}) - \delta(p - p_{(+)})\right] = 0. \quad (2.15)$$

Multiplying equation (2.15) first by 1 and then by p , and integrating the result over p , we readily obtain

$$\frac{\partial}{\partial\theta}(p_{(+)} - p_{(-)}) + \frac{\dot{\chi}}{2}\frac{\partial}{\partial x}(p_{(+)}^2 - p_{(-)}^2) = 0, \quad (2.16)$$

$$\frac{1}{2}\frac{\partial}{\partial\theta}(p_{(+)}^2 - p_{(-)}^2) + \frac{\dot{\chi}}{3}\frac{\partial}{\partial x}(p_{(+)}^3 - p_{(-)}^3) = -(p_{(+)} - p_{(-)})\left(\dot{\chi}x + \frac{\partial V}{\partial x} + \lambda\sqrt{\beta}\frac{\partial U}{\partial x}\right), \quad (2.17)$$

$$\frac{\partial^2 U}{\partial x^2} = -\mathcal{C}(p_{(+)} - p_{(-)}), \quad (2.18)$$

It is convenient to recast equations (2.16) - (2.18) in a more familiar form, widely used in hydrodynamics. Let us define

$$\rho = \int_{-\infty}^{\infty} dp f(x, p; \theta) = \mathcal{C}(p_{(+)} - p_{(-)}), \quad (2.19)$$

$$\rho v = \int_{-\infty}^{\infty} dp p f(x, p; \theta) = \frac{\mathcal{C}}{2}(p_{(+)}^2 - p_{(-)}^2), \quad (2.20)$$

where $\rho(x; \theta)$ and $v(x; \theta)$ are the density and the current velocity, respectively. From the two definitions above, it follows that

$$v = \frac{1}{2}(p_{(+)} + p_{(-)}). \quad (2.21)$$

Defining further the pressure $\mathcal{P}(x; \theta)$ and the heat flow $\mathcal{Q}(x; \theta)$ and using equation (2.21), we have

$$\mathcal{P} = \int_{-\infty}^{\infty} dp (p - v)^2 f(x, p; \theta) = \frac{1}{12\mathcal{C}^2}[\mathcal{C}(p_{(+)} - p_{(-)})]^3, \quad (2.22)$$

$$\mathcal{Q} = \int_{-\infty}^{\infty} dp (p - v)^3 f(x, p; \theta) = 0. \quad (2.23)$$

From equation (2.19), it follows that the pressure can be expressed as

$$\mathcal{P} = \frac{\mathcal{P}_0}{\hat{\rho}_0^3} \rho^3, \quad \frac{\mathcal{P}_0}{\hat{\rho}_0^3} = \frac{1}{12\mathcal{C}^2}. \quad (2.24)$$

In addition, it is straightforward to verify that

$$\frac{\mathcal{C}}{3}(p_{(+)}^3 - p_{(-)}^3) = \mathcal{P} + \rho v^2, \quad (2.25)$$

which provides a closure for the equations (2.16) and (2.17) governing the evolution of the moments (boundary curves). With all the above definitions and relations in hand, we can

write the system of hydrodynamic equations (supplemented by the Poisson equation) in the form

$$\frac{\partial \rho}{\partial \theta} + \dot{\chi} \frac{\partial}{\partial x}(\rho v) = 0, \quad (2.26)$$

$$\frac{\partial}{\partial \theta}(\rho v) + \dot{\chi} \frac{\partial}{\partial x}(\mathcal{P} + \rho v^2) = -\rho \left(\dot{\chi} x + \frac{\partial V}{\partial x} + \lambda \sqrt{\beta} \frac{\partial U}{\partial x} \right), \quad (2.27)$$

$$\frac{\partial^2 U}{\partial x^2} = -\rho. \quad (2.28)$$

Using the continuity equation (2.26), we cast the system of equations (2.26) - (2.28) in its final form

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial x}(\rho v) = 0, \quad (2.29)$$

$$\frac{\partial v}{\partial \tau} + v \frac{\partial v}{\partial x} + v_T^2 \frac{\partial}{\partial x}(\rho^2) = -x - \frac{\beta}{R} \frac{\partial V}{\partial x} - \lambda \frac{\beta \sqrt{\beta}}{R} \frac{\partial U}{\partial x}, \quad (2.30)$$

$$\frac{\partial^2 U}{\partial x^2} = -\rho, \quad (2.31)$$

which will be the starting point for the subsequent analysis. Here

$$v_T^2 = \frac{3\mathcal{P}_0}{2\hat{\rho}_0^3}, \quad (2.32)$$

is the normalized thermal speed-squared, and $\tau = \chi(\theta) + \tau_0$ is the new independent "time"-variable.

3 Renormalization Group Reduction of the Hydrodynamic Equations

Before proceeding further, we make an important remark. Let $F(\theta)$ be a periodic function of θ with period 2π . Noting that the phase advance $\chi(\theta)$ can be represented in the form

$$\chi(\theta) = \nu\theta + \chi_p(\theta) - \chi_p(\pi), \quad (3.1)$$

where ν is the betatron tune, and $\chi_p(\theta + 2\pi) = \chi_p(\theta)$, we can expand $F(\theta)$ regarded as a function of χ (respectively τ) in a Fourier series in the new variable χ (respectively τ) as follows

$$F(\theta) = \sum_{n=-\infty}^{\infty} \mathcal{A}_n \exp\left(in \frac{\chi}{\nu}\right). \quad (3.2)$$

The expansion coefficients \mathcal{A}_n are given by the expressions

$$\mathcal{A}_n = \frac{1}{2\pi\nu} \int_{-\pi\nu}^{\pi\nu} d\chi F(\theta) \exp\left(-in \frac{\chi}{\nu}\right). \quad (3.3)$$

Using the definition of the phase advance (3.1), and choosing $\tau_0 = \chi_p(\pi)$, we can rewrite the Fourier expansion (3.2) as

$$F(\theta) = \sum_{n=-\infty}^{\infty} \mathcal{B}_n \exp\left(in\frac{\tau}{\nu}\right), \quad (3.4)$$

where the expansion coefficients \mathcal{B}_n are given by the expressions

$$\mathcal{B}_n = \frac{R}{2\pi\nu} \int_{-\pi}^{\pi} d\theta \frac{F(\theta)}{\beta(\theta)} e^{-in\theta} \exp\left[-in\frac{\chi_p(\theta)}{\nu}\right]. \quad (3.5)$$

In what follows, we consider the case where the external potential V is zero, so that the set of equations to be analyzed acquires the form

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial x}(\rho v) = 0, \quad (3.6)$$

$$\frac{\partial v}{\partial \tau} + v \frac{\partial v}{\partial x} + v_T^2 \frac{\partial}{\partial x}(\rho^2) = -x - \lambda g(\tau) \frac{\partial U}{\partial x}, \quad (3.7)$$

$$\frac{\partial^2 U}{\partial x^2} = -\rho, \quad (3.8)$$

where

$$g(\tau) = \sum_{n=-\infty}^{\infty} a_n \exp\left(in\frac{\tau}{\nu}\right), \quad (3.9)$$

$$a_n = \frac{1}{2\pi\nu} \int_{-\pi}^{\pi} d\theta \sqrt{\beta(\theta)} e^{-in\theta} \exp\left[-in\frac{\chi_p(\theta)}{\nu}\right]. \quad (3.10)$$

Further, we assume that there exists a nontrivial solution to equations (3.6) - (3.8) in the interval $x \in (-x_{(-)}, x_{(+)})$, and that the sheet beam density is zero ($\rho = 0$) outside of the interval. We introduce also the ansatz

$$\rho(x; \tau) = \frac{1}{\mathcal{E}} + \epsilon R(x; \tau), \quad v(x; \tau) = \frac{x}{\mathcal{E}} \frac{d\mathcal{E}}{d\tau} + \epsilon u(x; \tau), \quad U(x; \tau) = -\frac{x^2}{2\mathcal{E}} + \epsilon \mathcal{U}(x; \tau), \quad (3.11)$$

where the envelope function $\mathcal{E}(\tau)$ is a solution to the equation

$$\frac{d^2 \mathcal{E}}{d\tau^2} + \mathcal{E} = \lambda g(\tau), \quad (3.12)$$

and therefore can be represented in explicit form as

$$\mathcal{E}(\tau) = \lambda a_0 + \lambda \sum_{n \neq 0} b_n e^{in\tau/\nu}, \quad b_n = \frac{a_n}{1 - n^2/\nu^2}. \quad (3.13)$$

It enables us to rewrite equations (3.6) - (3.8) as follows

$$\frac{\partial R}{\partial \tau} + \frac{1}{\mathcal{E}} \frac{\partial u}{\partial x} + \frac{\dot{\mathcal{E}}}{\mathcal{E}} \frac{\partial}{\partial x}(xR) + \epsilon \frac{\partial}{\partial x}(Ru) = 0, \quad (3.14)$$

$$\frac{\partial u}{\partial \tau} + \frac{\dot{\mathcal{E}}}{\mathcal{E}} \frac{\partial}{\partial x}(xu) + \frac{2v_T^2}{\mathcal{E}} \frac{\partial R}{\partial x} + \epsilon u \frac{\partial u}{\partial x} + \epsilon v_T^2 \frac{\partial}{\partial x}(R^2) = -\lambda g \frac{\partial \mathcal{U}}{\partial x}, \quad (3.15)$$

$$\frac{\partial^2 \mathcal{U}}{\partial x^2} = -R.$$

Next, we differentiate the first equation (3.14) of the above system with respect to τ and the second equation (3.15) with respect to x . After summing these up, we obtain

$$\begin{aligned} & \mathcal{E} \frac{\partial}{\partial \tau} \left(\mathcal{E} \frac{\partial R}{\partial \tau} \right) - 2v_T^2 \frac{\partial^2 R}{\partial x^2} + \lambda g \mathcal{E} R \\ &= \dot{\mathcal{E}} \frac{\partial^2}{\partial x^2}(xu) - \mathcal{E} \frac{\partial}{\partial \tau} \left[\dot{\mathcal{E}} \frac{\partial}{\partial x}(xR) \right] + \epsilon \mathcal{E} \frac{\partial^2}{\partial x^2} \left(\frac{u^2}{2} + v_T^2 R^2 \right) - \epsilon \mathcal{E} \frac{\partial}{\partial \tau} \left[\mathcal{E} \frac{\partial}{\partial x}(Ru) \right]. \end{aligned} \quad (3.16)$$

Equation (3.16) supplemented with equation (3.14) rewritten as

$$\mathcal{E} \frac{\partial R}{\partial \tau} + \frac{\partial u}{\partial x} + \dot{\mathcal{E}} \frac{\partial}{\partial x}(xR) + \epsilon \mathcal{E} \frac{\partial}{\partial x}(Ru) = 0, \quad (3.17)$$

comprises the basic system of equations for the analysis in the subsequent exposition.

In what follows, we consider the case of smooth focusing, where the time variation of $g(\tau)$ [$\mathcal{E}(\tau)$, respectively] can be neglected. In the next section, we will show that when the frequencies of all fundamental modes are sufficiently far from a parametric resonance, this assumption holds true to second order in the perturbation parameter ϵ even in the case when such time variation is present. Thus, the basic equations (3.16) and (3.17) acquire the form

$$\mathcal{E}_0^2 \frac{\partial^2 R}{\partial \tau^2} - 2v_T^2 \frac{\partial^2 R}{\partial x^2} + \mathcal{E}_0^2 R = \epsilon \mathcal{E}_0 \frac{\partial^2}{\partial x^2} \left(\frac{u^2}{2} + v_T^2 R^2 \right) - \epsilon \mathcal{E}_0^2 \frac{\partial^2}{\partial \tau \partial x}(Ru), \quad (3.18)$$

$$\mathcal{E}_0 \frac{\partial R}{\partial \tau} + \frac{\partial u}{\partial x} + \epsilon \mathcal{E}_0 \frac{\partial}{\partial x}(Ru) = 0, \quad (3.19)$$

where

$$\mathcal{E}_0 = \lambda a_0. \quad (3.20)$$

Let us further assume that the actual dependence of R and u on the independent variables is given by

$$R = R(x, \xi; \tau), \quad u = u(x, \xi; \tau), \quad (3.21)$$

where

$$\xi = \epsilon x \quad (3.22)$$

is a slow spatial variable. Therefore, the basic equations (3.18) and (3.19) can be rewritten as

$$\mathcal{E}_0^2 \frac{\partial^2 R}{\partial \tau^2} - 2v_T^2 \left(\frac{\partial^2}{\partial x^2} + 2\epsilon \frac{\partial^2}{\partial x \partial \xi} + \epsilon^2 \frac{\partial^2}{\partial \xi^2} \right) R + \mathcal{E}_0^2 R$$

$$= \epsilon \mathcal{E}_0 \left(\frac{\partial^2}{\partial x^2} + 2\epsilon \frac{\partial^2}{\partial x \partial \xi} + \epsilon^2 \frac{\partial^2}{\partial \xi^2} \right) \left(\frac{u^2}{2} + v_T^2 R^2 \right) - \epsilon \mathcal{E}_0^2 \left(\frac{\partial^2}{\partial \tau \partial x} + \epsilon \frac{\partial^2}{\partial \tau \partial \xi} \right) (Ru), \quad (3.23)$$

$$\mathcal{E}_0 \frac{\partial R}{\partial \tau} + \frac{\partial u}{\partial x} + \epsilon \frac{\partial u}{\partial \xi} + \epsilon \mathcal{E}_0 \left(\frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi} \right) (Ru) = 0, \quad (3.24)$$

Following the basic idea of the RG method, we represent the solution to equations (3.23) and (3.24) in the form of a standard perturbation expansion [7] in the formal small parameter ϵ as

$$R = \sum_{k=0}^{\infty} \epsilon^k R_k, \quad u = \sum_{k=0}^{\infty} \epsilon^k u_k, \quad (3.25)$$

The zeroth-order equations (3.23) and (3.24) read as

$$\lambda^2 a_0^2 \frac{\partial^2 R_0}{\partial \tau^2} - 2v_T^2 \frac{\partial^2 R_0}{\partial x^2} + \lambda^2 a_0^2 R_0 = 0, \quad (3.26)$$

$$\lambda a_0 \frac{\partial R_0}{\partial \tau} + \frac{\partial u_0}{\partial x} = 0. \quad (3.27)$$

Their solutions can be found in a straightforward manner to be

$$R_0(x, \xi; \tau) = \sum_{m \neq 0} \left[A_m(\xi) e^{iz_m^{(+)}} + B_m(\xi) e^{iz_m^{(-)}} \right], \quad (3.28)$$

$$u_0(x, \xi; \tau) = -\frac{2\pi}{\sigma^2} \sum_{m \neq 0} \frac{\omega_m}{m} \left[A_m(\xi) e^{iz_m^{(+)}} - B_m(\xi) e^{iz_m^{(-)}} \right]. \quad (3.29)$$

Here A_m and B_m are constant complex amplitudes (to this end dependent on the slow variable ξ only) of the backward and the forward wave solution to equation (3.26), respectively. These will be the subject of renormalization at the final step of the renormalization group procedure resulting in RG equations governing their slow evolution. Furthermore,

$$z_m^{(\pm)}(x; \tau) = \omega_m \tau \pm m\sigma x, \quad \sigma = \frac{2\pi}{x_{(+)} + x_{(-)}}, \quad \lambda a_0 = \frac{2\pi}{\sigma}. \quad (3.30)$$

The discrete mode frequencies ω_m are determined from the dispersion relation

$$\omega_m^2 = 1 + K^2 m^2, \quad K = \frac{v_T^2 \sigma^4}{2\pi^2}. \quad (3.31)$$

It can be easily verified that the above choice of parameters leads to

$$\int_{-x_{(-)}}^{x_{(+)}} dx R_0(x; \theta) = 0, \quad (3.32)$$

which means that linear perturbation to the uniform density $\rho_0 = \mathcal{E}_0^{-1}$ average to zero and do not affect the normalization properties on the interval $(x^{(-)}, x^{(+)})$. In addition, the following conventions and notations

$$\omega_{-m} = -\omega_m, \quad A_{-m} = A_m^*, \quad B_{-m} = B_m^* \quad (3.33)$$

have been introduced.

The first order perturbation equation for R_1 reads as

$$\begin{aligned} \frac{\partial^2 R_1}{\partial \tau^2} - \frac{K^2}{\sigma^2} \frac{\partial^2 R_1}{\partial x^2} + R_1 &= \frac{2iK^2}{\sigma} \sum_{m \neq 0} m \left(\frac{\partial A_m}{\partial \xi} e^{iz_m^{(+)}} - \frac{\partial B_m}{\partial \xi} e^{iz_m^{(-)}} \right) \\ &- \frac{\pi}{\sigma} \sum_{m,n \neq 0} \left[\gamma_{mn}^{(+)} A_m A_n e^{i(z_m^{(+)} + z_n^{(+)})} + 2\gamma_{mn}^{(-)} A_m B_n e^{i(z_m^{(+)} + z_n^{(-)})} + \gamma_{mn}^{(+)} B_m B_n e^{i(z_m^{(-)} + z_n^{(-)})} \right], \end{aligned} \quad (3.34)$$

where

$$\gamma_{mn}^{(\pm)} = (m \pm n) \left[(\omega_m + \omega_n) \left(\frac{\omega_m}{m} \pm \frac{\omega_n}{n} \right) + (m \pm n) \left(K^2 \pm \frac{\omega_m \omega_n}{mn} \right) \right]. \quad (3.35)$$

The solution for R_1 is readily obtained to be

$$\begin{aligned} R_1(x, \xi; \tau) &= \frac{K^2 \tau}{\sigma} \sum_{m \neq 0} \frac{m}{\omega_m} \left(\frac{\partial A_m}{\partial \xi} e^{iz_m^{(+)}} - \frac{\partial B_m}{\partial \xi} e^{iz_m^{(-)}} \right) \\ &- \frac{\pi}{\sigma} \sum_{m,n \neq 0} \left[\alpha_{mn}^{(+)} A_m A_n e^{i(z_m^{(+)} + z_n^{(+)})} + 2\alpha_{mn}^{(-)} A_m B_n e^{i(z_m^{(+)} + z_n^{(-)})} + \alpha_{mn}^{(+)} B_m B_n e^{i(z_m^{(-)} + z_n^{(-)})} \right], \end{aligned} \quad (3.36)$$

where

$$\alpha_{mn}^{(\pm)} = \frac{\gamma_{mn}^{(\pm)}}{\mathcal{D}_{mn}^{(\pm)}}, \quad (3.37)$$

$$\mathcal{D}_{mn}^{(\pm)} = 1 - (\omega_m + \omega_n)^2 + K^2(m \pm n)^2. \quad (3.38)$$

Note that the (infinite dimensional) matrices $\hat{\alpha}^{(\pm)}$ are symmetric, i.e.

$$\alpha_{mn}^{(\pm)} = \alpha_{nm}^{(\pm)}, \quad \alpha_{m,\mp m}^{(\pm)} = 0. \quad (3.39)$$

Having determined $R_1(x; \tau)$, the first-order current velocity $u_1(x; \tau)$ can be found in a straightforward manner. The result is

$$\begin{aligned} u_1(x, \xi; \tau) &= -\frac{2\pi K^2 \tau}{\sigma^3} \sum_{m \neq 0} \left(\frac{\partial A_m}{\partial \xi} e^{iz_m^{(+)}} + \frac{\partial B_m}{\partial \xi} e^{iz_m^{(-)}} \right) \\ &- \frac{2\pi i}{\sigma^3} \sum_{m \neq 0} \frac{1}{m^2 \omega_m} \left(\frac{\partial A_m}{\partial \xi} e^{iz_m^{(+)}} + \frac{\partial B_m}{\partial \xi} e^{iz_m^{(-)}} \right) \\ &+ \frac{2\pi^2}{\sigma^3} \sum_{m,n \neq 0} \left[\beta_{mn}^{(+)} A_m A_n e^{i(z_m^{(+)} + z_n^{(+)})} + 2\beta_{mn}^{(-)} A_m B_n e^{i(z_m^{(+)} + z_n^{(-)})} - \beta_{mn}^{(+)} B_m B_n e^{i(z_m^{(-)} + z_n^{(-)})} \right], \end{aligned} \quad (3.40)$$

where

$$\beta_{mn}^{(\pm)} = \frac{\omega_m}{m} \pm \frac{\omega_n}{n} + \alpha_{mn}^{(\pm)} \frac{\omega_m + \omega_n}{m \pm n}, \quad \beta_{m,-m}^{(+)} = 0. \quad (3.41)$$

Note that the (infinite dimensional) matrix $\hat{\beta}^{(+)}$ possesses the same symmetry properties as those displayed by equation (3.39) possessed by the matrix $\hat{\alpha}^{(+)}$, while $\hat{\beta}^{(-)}$ is antisymmetric (and evidently $\beta_{mm}^{(-)} = 0$).

In second order, the equation for $R_2(x, \xi; \tau)$ acquires a form similar to that of equation (3.34). It is important to emphasize that this assertion holds true in every subsequent order. Each entry on the right-hand-sides of the corresponding equations can be calculated explicitly utilizing the already determined quantities from the previous orders. The right-hand-side of the equation for $R_2(x, \xi; \tau)$ contains terms which yield oscillating contributions with constant amplitudes to the solution for $R_2(x, \xi; \tau)$. Apart from these, there are resonant terms (proportional to $e^{iz_m^{(\pm)}(x; \tau)}$) leading to a secular contribution. To complete the renormalization group reduction of the hydrodynamic equations, we select these particular resonant second-order terms on the right-hand-side of the equation determining $R_2(x, \xi; \tau)$. The latter can be written as

$$\begin{aligned} \frac{\partial^2 R_2}{\partial \tau^2} - \frac{K^2}{\sigma^2} \frac{\partial^2 R_2}{\partial x^2} + R_2 &= \frac{K^2}{\sigma^2} \sum_{m \neq 0} \left(\frac{\partial^2 A_m}{\partial \xi^2} e^{iz_m^{(+)}} + \frac{\partial^2 B_m}{\partial \xi^2} e^{iz_m^{(-)}} \right) \\ &+ \frac{2iK^4\tau}{\sigma^2} \sum_{m \neq 0} \frac{m^2}{\omega_m} \left(\frac{\partial^2 A_m}{\partial \xi^2} e^{iz_m^{(+)}} + \frac{\partial^2 B_m}{\partial \xi^2} e^{iz_m^{(-)}} \right) \\ &+ \frac{4\pi^2}{\sigma^2} \sum_{m, n \neq 0} \left[\left(\Gamma_{mn}^{(+)} |A_n|^2 + \Gamma_{mn}^{(-)} |B_n|^2 \right) A_m e^{iz_m^{(+)}} + \left(\Gamma_{mn}^{(-)} |A_n|^2 + \Gamma_{mn}^{(+)} |B_n|^2 \right) B_m e^{iz_m^{(-)}} \right], \end{aligned} \quad (3.42)$$

where

$$\Gamma_{mn}^{(+)} = m^2 \left[\beta_{mn}^{(+)} \left(\frac{\omega_m}{m} + \frac{\omega_n}{n} \right) + \alpha_{mn}^{(+)} \left(K^2 + \frac{\omega_m \omega_n}{mn} \right) \right] \left(1 - \frac{\delta_{mn}}{2} \right), \quad (3.43)$$

$$\Gamma_{mn}^{(-)} = m^2 \left[\beta_{mn}^{(-)} \left(\frac{\omega_m}{m} - \frac{\omega_n}{n} \right) + \alpha_{mn}^{(-)} \left(K^2 - \frac{\omega_m \omega_n}{mn} \right) \right], \quad (3.44)$$

Some straightforward algebra yields the solution for $R_2(x, \xi; \tau)$ in the form

$$\begin{aligned} R_2(x, \xi; \tau) &= \frac{K^4\tau^2}{2\sigma^2} \sum_{m \neq 0} \frac{m^2}{\omega_m^2} \left(\frac{\partial^2 A_m}{\partial \xi^2} e^{iz_m^{(+)}} + \frac{\partial^2 B_m}{\partial \xi^2} e^{iz_m^{(-)}} \right) \\ &+ \frac{K^2\tau}{2i\sigma^2} \sum_{m \neq 0} \frac{1}{\omega_m^3} \left(\frac{\partial^2 A_m}{\partial \xi^2} e^{iz_m^{(+)}} + \frac{\partial^2 B_m}{\partial \xi^2} e^{iz_m^{(-)}} \right) \\ &+ \frac{2\pi^2\tau}{i\sigma^2} \sum_{m, n \neq 0} \frac{1}{\omega_m} \left[\left(\Gamma_{mn}^{(+)} |A_n|^2 + \Gamma_{mn}^{(-)} |B_n|^2 \right) A_m e^{iz_m^{(+)}} + \left(\Gamma_{mn}^{(-)} |A_n|^2 + \Gamma_{mn}^{(+)} |B_n|^2 \right) B_m e^{iz_m^{(-)}} \right], \end{aligned} \quad (3.45)$$

where non-secular oscillating terms have not been written out in full explicitly. The final step is to collect the terms proportional to the fundamental modes $e^{iz_m^{(+)}}$ and $e^{iz_m^{(-)}}$ in all

orders and renormalize the amplitudes A_m and B_m . As a result one obtains the following RG equations

$$2i\omega_m \frac{\partial A_m}{\partial \tau} - 2im \frac{K^2}{\sigma} \frac{\partial A_m}{\partial x} = \frac{K^2}{\sigma^2 \omega_m^2} \frac{\partial^2 A_m}{\partial x^2} + \frac{4\pi^2}{\sigma^2} A_m \sum_{n \neq 0} \left(\Gamma_{mn}^{(+)} |A_n|^2 + \Gamma_{mn}^{(-)} |B_n|^2 \right), \quad (3.46)$$

$$2i\omega_m \frac{\partial B_m}{\partial \tau} + 2im \frac{K^2}{\sigma} \frac{\partial B_m}{\partial x} = \frac{K^2}{\sigma^2 \omega_m^2} \frac{\partial^2 B_m}{\partial x^2} + \frac{4\pi^2}{\sigma^2} B_m \sum_{n \neq 0} \left(\Gamma_{mn}^{(-)} |A_n|^2 + \Gamma_{mn}^{(+)} |B_n|^2 \right), \quad (3.47)$$

4 The Parametric Resonance

Let us now address the system of equations (3.16) and (3.17). Without loss of generality, we assume that the time variation of $g(\tau)$ can be treated as a second-order perturbation (which is usually the case), that is

$$g(\tau) = a_0 + \epsilon^2 \sum_{n \neq 0} a_n e^{in\tau/\nu}. \quad (4.1)$$

The same holds true for the envelope function $\mathcal{E}(\tau)$

$$\mathcal{E}(\tau) = \lambda a_0 + \epsilon^2 \lambda \sum_{n \neq 0} b_n e^{in\tau/\nu}, \quad b_n = \frac{a_n}{1 - n^2/\nu^2}. \quad (4.2)$$

Thus, the basic equations (3.16) and (3.17) can be rewritten in the form

$$\begin{aligned} & \mathcal{E} \frac{\partial}{\partial \tau} \left(\mathcal{E} \frac{\partial R}{\partial \tau} \right) - 2v_T^2 \frac{\partial^2 R}{\partial x^2} + \lambda g \mathcal{E} R \\ &= \epsilon^2 \dot{\mathcal{E}}_2 \frac{\partial^2}{\partial x^2} (xu) - \epsilon^2 \mathcal{E} \frac{\partial}{\partial \tau} \left[\dot{\mathcal{E}}_2 \frac{\partial}{\partial x} (xR) \right] + \epsilon \mathcal{E} \frac{\partial^2}{\partial x^2} \left(\frac{u^2}{2} + v_T^2 R^2 \right) - \epsilon \mathcal{E} \frac{\partial}{\partial \tau} \left[\mathcal{E} \frac{\partial}{\partial x} (Ru) \right]. \end{aligned} \quad (4.3)$$

$$\mathcal{E} \frac{\partial R}{\partial \tau} + \frac{\partial u}{\partial x} + \epsilon^2 \dot{\mathcal{E}}_2 \frac{\partial}{\partial x} (xR) + \epsilon \mathcal{E} \frac{\partial}{\partial x} (Ru) = 0, \quad (4.4)$$

Let us reiterate that the assumption concerning the second order of magnitude in ϵ of the time variation of $g(\tau)$ and $\mathcal{E}(\tau)$ does not restrict the generality of the subsequent analysis. If this variation were of first order in ϵ , the proper perturbation parameter to use would be $\sqrt{\epsilon}$ instead of ϵ . In addition, the variables R and u have to be rescaled accordingly

$$R \longrightarrow \frac{R}{\sqrt{\epsilon}}, \quad u \longrightarrow \frac{u}{\sqrt{\epsilon}}. \quad (4.5)$$

In terms of the ansatz (3.21) and (3.22), equations (4.3) and (4.4) become

$$\begin{aligned} & \mathcal{E} \frac{\partial}{\partial \tau} \left(\mathcal{E} \frac{\partial R}{\partial \tau} \right) - 2v_T^2 \left(\frac{\partial^2}{\partial x^2} + 2\epsilon \frac{\partial^2}{\partial x \partial \xi} + \epsilon^2 \frac{\partial^2}{\partial \xi^2} \right) R + \lambda g \mathcal{E} R \\ &= \epsilon \dot{\mathcal{E}}_2 \left(\frac{\partial^2}{\partial x^2} + 2\epsilon \frac{\partial^2}{\partial x \partial \xi} + \epsilon^2 \frac{\partial^2}{\partial \xi^2} \right) (\xi u) - \epsilon \mathcal{E} \frac{\partial}{\partial \tau} \left[\dot{\mathcal{E}}_2 \left(\frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi} \right) (\xi R) \right] \end{aligned}$$

$$+ \epsilon \mathcal{E} \left(\frac{\partial^2}{\partial x^2} + 2\epsilon \frac{\partial^2}{\partial x \partial \xi} + \epsilon^2 \frac{\partial^2}{\partial \xi^2} \right) \left(\frac{u^2}{2} + v_T^2 R^2 \right) - \epsilon \mathcal{E} \frac{\partial}{\partial \tau} \left[\mathcal{E} \left(\frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi} \right) (Ru) \right], \quad (4.6)$$

$$\mathcal{E} \frac{\partial R}{\partial \tau} + \frac{\partial u}{\partial x} + \epsilon \frac{\partial u}{\partial \xi} + \epsilon \dot{\mathcal{E}}_2 \left(\frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi} \right) (\xi R) + \epsilon \mathcal{E} \left(\frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi} \right) (Ru) = 0. \quad (4.7)$$

Although the general case can be in principle treated through more labour-intensive manipulations, for the sake of simplicity in what follows, we select a particular mode with mode number m . The zeroth-order solution can be written in the form

$$R_0(x, \xi; \tau) = A(\xi) e^{iz^{(+)}(x, \tau)} + B(\xi) e^{iz^{(-)}(x, \tau)} + c.c., \quad (4.8)$$

$$u_0(x, \xi; \tau) = -\frac{2\pi\omega}{m\sigma^2} [A(\xi) e^{iz^{(+)}(x, \tau)} - B(\xi) e^{iz^{(-)}(x, \tau)}] + c.c., \quad (4.9)$$

where again

$$z^{(\pm)}(x, \tau) = \omega\tau \pm m\sigma x, \quad \omega^2 = 1 + K^2 m^2.$$

The particular mode is chosen such that an exact parametric resonance (if possible)

$$2\omega = \frac{n_0}{\nu} \quad (4.10)$$

occurs for some integer n_0 . The first order perturbation equation for R_1 can be written in the form

$$\begin{aligned} \frac{\partial^2 R_1}{\partial \tau^2} - \frac{K^2}{\sigma^2} \frac{\partial^2 R_1}{\partial x^2} + R_1 &= \frac{2imK^2}{\sigma} \left(\frac{\partial A}{\partial \xi} e^{iz^{(+)}(x, \tau)} - \frac{\partial B}{\partial \xi} e^{iz^{(-)}(x, \tau)} \right) \\ &+ \frac{im\sigma^2}{2\pi} \xi \lambda \sum'_{n \neq 0} \frac{n}{\nu} \left(2\omega + \frac{n}{\nu} \right) b_n e^{in\tau/\nu} \left(A e^{iz^{(+)}(x, \tau)} - B e^{iz^{(-)}(x, \tau)} \right) \\ &- \frac{4\pi}{\sigma} (4\omega^2 - 1) \left(A^2 e^{2iz^{(+)}(x, \tau)} + B^2 e^{2iz^{(-)}(x, \tau)} \right) + \frac{8\pi}{\sigma} AB^* e^{i(z^{(+)} - z^{(-)})} + c.c.. \end{aligned} \quad (4.11)$$

Here \sum' implies exclusion of the harmonic with $n = n_0$ from the sum. It is important to note that terms giving rise to secular contribution due to the parametric resonance vanish identically in the first order. The solution for R_1 can be written explicitly as

$$\begin{aligned} R_1(x, \xi; \tau) &= \frac{mK^2}{\sigma\omega} \tau \left(\frac{\partial A}{\partial \xi} e^{iz^{(+)}(x, \tau)} - \frac{\partial B}{\partial \xi} e^{iz^{(-)}(x, \tau)} \right) - \frac{im\sigma^2}{2\pi} \xi \lambda \sum'_{n \neq 0} b_n e^{in\tau/\nu} \left(A e^{iz^{(+)}(x, \tau)} - B e^{iz^{(-)}(x, \tau)} \right) \\ &+ \frac{4\pi}{3\sigma} (4\omega^2 - 1) \left(A^2 e^{2iz^{(+)}(x, \tau)} + B^2 e^{2iz^{(-)}(x, \tau)} \right) + \frac{8\pi}{\sigma(4\omega^2 - 3)} AB^* e^{i(z^{(+)} - z^{(-)})} + c.c.. \end{aligned} \quad (4.12)$$

Straightforward calculations yield the solution for u_1

$$u_1(x, \xi; \tau) = -\frac{2\pi i}{m^2 \sigma^3 \omega} \left(\frac{\partial A}{\partial \xi} e^{iz^{(+)}(x, \tau)} + \frac{\partial B}{\partial \xi} e^{iz^{(-)}(x, \tau)} \right) - \frac{2\pi K^2}{\sigma^3} \tau \left(\frac{\partial A}{\partial \xi} e^{iz^{(+)}(x, \tau)} + \frac{\partial B}{\partial \xi} e^{iz^{(-)}(x, \tau)} \right)$$

$$\begin{aligned}
& -\frac{i\lambda n_0}{\nu}\xi\left(b_{n_0}e^{in_0\tau/\nu}-b_{n_0}^*e^{-in_0\tau/\nu}\right)\left(Ae^{iz^{(+)}}+Be^{iz^{(-)}}\right) \\
& +i\omega\xi\lambda\sum'_{n\neq 0}b_ne^{in\tau/\nu}\left(Ae^{iz^{(+)}}+Be^{iz^{(-)}}\right)-\frac{8\pi^2}{3m\sigma^3}\omega\left(4\omega^2-5/2\right)\left(A^2e^{2iz^{(+)}}-B^2e^{2iz^{(-)}}\right)+c.c..
\end{aligned} \tag{4.13}$$

In second order, we take into account terms providing secular contribution to the solution only. Thus, we write the equation for R_2 as

$$\begin{aligned}
& \frac{\partial^2 R_2}{\partial \tau^2}-\frac{K^2}{\sigma^2}\frac{\partial^2 R_2}{\partial x^2}+R_2=\frac{2iK^4m^2}{\omega\sigma^2}\tau\left(\frac{\partial^2 A}{\partial \xi^2}e^{iz^{(+)}}+\frac{\partial^2 B}{\partial \xi^2}e^{iz^{(-)}}\right) \\
& +\frac{\lambda\sigma}{\pi}b_{n_0}(1-\omega^2)\left[\left(\xi\frac{\partial}{\partial \xi}-1\right)B^*e^{iz^{(+)}}+\left(\xi\frac{\partial}{\partial \xi}-1\right)A^*e^{iz^{(-)}}\right] \\
& +\frac{K^2}{\sigma^2}\left(\frac{\partial^2 A}{\partial \xi^2}e^{iz^{(+)}}+\frac{\partial^2 B}{\partial \xi^2}e^{iz^{(-)}}\right)+\frac{2\lambda^2\sigma^2}{\pi^2}\omega^2m^2\sigma^2|b_{n_0}|^2\xi^2\left(Ae^{iz^{(+)}}+Be^{iz^{(-)}}\right) \\
& -\frac{8\pi^2}{3\sigma^2}\left(16\omega^4-11\omega^2+1\right)\left(|A|^2Ae^{iz^{(+)}}+|B|^2Be^{iz^{(-)}}\right) \\
& +\frac{16\pi^2}{\sigma^2(4\omega^2-3)}\left(|B|^2Ae^{iz^{(+)}}+|A|^2Be^{iz^{(-)}}\right)+c.c..
\end{aligned} \tag{4.14}$$

Similar to equation (3.45), we obtain in a straightforward manner

$$\begin{aligned}
R_2(x,\xi;\tau) & =\frac{K^4m^2}{2\omega^2\sigma^2}\tau^2\left(\frac{\partial^2 A}{\partial \xi^2}e^{iz^{(+)}}+\frac{\partial^2 B}{\partial \xi^2}e^{iz^{(-)}}\right)+\frac{K^2\tau}{2i\omega^3\sigma^2}\left(\frac{\partial^2 A}{\partial \xi^2}e^{iz^{(+)}}+\frac{\partial^2 B}{\partial \xi^2}e^{iz^{(-)}}\right) \\
& +\frac{\lambda\sigma\tau}{2\pi i\omega}b_{n_0}(1-\omega^2)\left[\left(\xi\frac{\partial}{\partial \xi}-1\right)B^*e^{iz^{(+)}}+\left(\xi\frac{\partial}{\partial \xi}-1\right)A^*e^{iz^{(-)}}\right] \\
& +\frac{2\lambda^2\sigma^2\tau}{2i\pi^2\omega}\omega^2m^2\sigma^2|b_{n_0}|^2\xi^2\left(Ae^{iz^{(+)}}+Be^{iz^{(-)}}\right) \\
& -\frac{\tau}{2i\omega}\frac{8\pi^2}{3\sigma^2}\left(16\omega^4-11\omega^2+1\right)\left(|A|^2Ae^{iz^{(+)}}+|B|^2Be^{iz^{(-)}}\right) \\
& +\frac{\tau}{2i\omega}\frac{16\pi^2}{\sigma^2(4\omega^2-3)}\left(|B|^2Ae^{iz^{(+)}}+|A|^2Be^{iz^{(-)}}\right)+c.c..
\end{aligned} \tag{4.15}$$

Collecting once again the zeroth-order term together with all secular terms in higher orders and renormalizing the amplitudes A and B , we obtain the RG equations

$$2i\omega \frac{\partial A}{\partial \tau} - 2im \frac{K^2}{\sigma} \frac{\partial A}{\partial x} = \frac{K^2}{\sigma^2 \omega^2} \frac{\partial^2 A}{\partial x^2} + \frac{\lambda \sigma}{\pi} b_{n_0} (1 - \omega^2) \left(x \frac{\partial}{\partial x} - 1 \right) B^* \\ + \frac{2\lambda^2 \sigma^4}{\pi^2} m^2 \omega^2 |b_{n_0}|^2 x^2 A - \frac{8\pi^2}{3\sigma^2} (16\omega^4 - 11\omega^2 + 1) |A|^2 A + \frac{16\pi^2}{\sigma^2 (4\omega^2 - 3)} |B|^2 A, \quad (4.16)$$

$$2i\omega \frac{\partial B}{\partial \tau} + 2im \frac{K^2}{\sigma} \frac{\partial B}{\partial x} = \frac{K^2}{\sigma^2 \omega^2} \frac{\partial^2 B}{\partial x^2} + \frac{\lambda \sigma}{\pi} b_{n_0} (1 - \omega^2) \left(x \frac{\partial}{\partial x} - 1 \right) A^* \\ + \frac{2\lambda^2 \sigma^4}{\pi^2} m^2 \omega^2 |b_{n_0}|^2 x^2 B - \frac{8\pi^2}{3\sigma^2} (16\omega^4 - 11\omega^2 + 1) |B|^2 B + \frac{16\pi^2}{\sigma^2 (4\omega^2 - 3)} |A|^2 B. \quad (4.17)$$

5 The Nonlinear Schrodinger Equation for a Single Mode

Equations (3.46) and (3.47) represent a system of coupled nonlinear Schrodinger equations for the mode amplitudes A_m and B_m . Neglecting the contribution from modes with $n \neq \pm m$ and introducing a new amplitude $\tilde{B}_m(x; \tau) = B_m(-x; \tau)$, for the single mode amplitudes A_m and B_m , we obtain the equations

$$i \frac{\partial A}{\partial \tau} - \frac{\partial^2 A}{\partial \zeta^2} - \frac{8\pi^2}{\omega \sigma^2 (4\omega^2 - 3)} (-G|A|^2 + |B|^2) A = 0, \quad (5.1)$$

$$i \frac{\partial B}{\partial \tau} - \frac{\partial^2 B}{\partial \zeta^2} - \frac{8\pi^2}{\omega \sigma^2 (4\omega^2 - 3)} (|A|^2 - G|B|^2) B = 0, \quad (5.2)$$

where

$$\zeta = \sqrt{2\omega} \left(mK\tau + \frac{\omega\sigma}{K} \right), \quad G = \frac{4\omega^2 - 3}{6} (16\omega^4 - 11\omega^2 + 1), \quad (5.3)$$

the index m and the tilde sign over the new amplitude \tilde{B} has been omitted. Clearly enough, equations (5.1) and (5.2) follow directly from the system (4.16) - (4.17) if b_{n_0} is set to zero. This implies that in the case where a parametric resonance does not occur in the original system, the smooth approximation is valid to second order in the formal perturbation parameter. Moreover, it is straightforward to verify that G is always positive. The system of two coupled nonlinear Schrodinger equations (5.1) and (5.2) is in general non integrable. Its integrability is proven by Manakov [8] only in the simplest case of $G = -1$.

If one of the amplitudes (B or A) in its capacity of being a particular solution to the corresponding nonlinear Schrodinger equation is identically zero, the equation for the other amplitude (say A) becomes

$$i \frac{\partial A}{\partial \tau} - \frac{\partial^2 A}{\partial \zeta^2} + \Gamma |A|^2 A = 0, \quad (5.4)$$

where

$$\Gamma = \frac{4\pi^2}{3\omega\sigma^2}(16\omega^4 - 11\omega^2 + 1). \quad (5.5)$$

In nonlinear optics equation (5.4) is known to describe the formation and evolution of the so-called *dark solitons* [9]. In the case of charged particle beams these correspond to the formation of *holes* or *cavitons* in the beam. The solution to equation (5.4) can be written as

$$A(\zeta; \tau) = \frac{r(\xi)}{\sqrt{\Gamma}} e^{i[n\tau + \theta(\xi)]}, \quad \xi = \zeta - c\tau, \quad (5.6)$$

where

$$r(\xi) = \sqrt{n - 2a^2 \text{sech}^2(a\xi)}, \quad \theta(\xi) = \arctan \left[\frac{2a}{c} \tanh(a\xi) \right], \quad (5.7)$$

for all c and

$$a = \frac{1}{2} \sqrt{2n - c^2}, \quad (5.8)$$

provided $n > c^2/2$.

Let us now examine equations (4.16) and (4.17). In the cold-beam limit $v_T \rightarrow 0$ (or equivalently $\omega \rightarrow 1$) the second term on their right-hand-sides can be neglected as compared to the other terms. Therefore, we can write

$$\begin{aligned} 2i\omega \frac{\partial \tilde{A}}{\partial \tau} &= \frac{K^2}{\sigma^2 \omega^2} \frac{\partial^2 \tilde{A}}{\partial x^2} + \frac{2\lambda^2 \sigma^4}{\pi^2} m^2 \omega^2 |b_{n_0}|^2 x^2 \tilde{A} \\ &- \frac{8\pi^2}{3\sigma^2} (16\omega^4 - 11\omega^2 + 1) |\tilde{A}|^2 \tilde{A} + \frac{16\pi^2}{\sigma^2 (4\omega^2 - 3)} |\tilde{B}|^2 \tilde{A}, \end{aligned} \quad (5.9)$$

$$\begin{aligned} 2i\omega \frac{\partial \tilde{B}}{\partial \tau} &= \frac{K^2}{\sigma^2 \omega^2} \frac{\partial^2 \tilde{B}}{\partial x^2} + \frac{2\lambda^2 \sigma^4}{\pi^2} m^2 \omega^2 |b_{n_0}|^2 x^2 \tilde{B} \\ &- \frac{8\pi^2}{3\sigma^2} (16\omega^4 - 11\omega^2 + 1) |\tilde{B}|^2 \tilde{B} + \frac{16\pi^2}{\sigma^2 (4\omega^2 - 3)} |\tilde{A}|^2 \tilde{B}, \end{aligned} \quad (5.10)$$

where the local gauge transformation

$$\tilde{A} = A \exp \left[im\omega \left(\frac{mK^2}{2} \tau + \sigma\omega x \right) \right], \quad \tilde{B} = B \exp \left[im\omega \left(\frac{mK^2}{2} \tau - \sigma\omega x \right) \right], \quad (5.11)$$

has been performed on the amplitudes A and B . Equations (5.9) and (5.10) represent a system of two coupled Gross-Pitaevskii equations, known to govern the formation of condensate structures in atomic gases confined in magnetic traps. In addition, superfluidity in helium as well as patterns in the gas of paraexcitons in semiconductors are considered as a possible manifestation of Bose-Einstein condensation. It is remarkable that under certain conditions similar phenomenon can be observed in space-charge dominated beams.

6 Concluding Remarks

The analysis performed in the present paper is based on the Vlasov-Maxwell equations for the self-consistent evolution of the beam distribution function and the electromagnetic fields. We considered the propagation of an intense beam through a periodic focusing lattice. It has been proven that a special class of solutions to the Vlasov-Maxwell equations exists, which defines a uniform phase-space density inside of simply connected boundary curves. These solutions provide an exact closure to the hierarchy of moments resulting in a set of hydrodynamic equations for the beam density (continuity equation) and for the current velocity. The latter is characterized by a triple adiabatic pressure law. The major drawback of the model discussed here is that due to the constancy of the distribution function inside of the evolving region of phase space, it fails to take into account an important feature such as the well-known Landau damping. Inclusion of the Landau damping in the description can be achieved correctly only by direct analysis of the Vlasov-Maxwell system [10].

Further, we studied first the case when the smooth focusing approximation applies. Based on the RG method, a system of coupled nonlinear Schrodinger equations has been derived for the slowly varying amplitudes of interacting beam-density waves. Under the approximation of an isolated mode neglecting the effect of the rest of the modes, this system reduces to two coupled nonlinear Schrodinger equations for the amplitudes of the forward and of the backward wave. The particular solution to the latter system asserting that the amplitude of either of the waves (the forward or the backward) can vanish identically, leads to a single nonlinear Schrodinger equation for the other amplitude. The latter is characterized by a repulsive nonlinearity and therefore, it describes the evolution of *holes* in intense charged particle beams.

The analysis of periodic focusing clearly showed that the results obtained in the case of smooth focusing remain unchanged up to second order in the perturbation parameter if a parametric resonance between a particular mode and an appropriate Fourier harmonic of the external focusing does not occur. If however, an exact parametric resonance takes place, it was shown that the evolution of the wave amplitudes of the resonant mode in the cold-beam limit is described by a system of coupled Gross-Pitaevskii equations. Quite remarkably, it was found that there exist a possibility of formation of density condensates in space-charge dominated beams.

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Appendix

In order to prove that the expression (2.14) represents the solution of the Vlasov equation (2.12), we write the equations for the characteristics of the latter in the form

$$\frac{d\theta}{1} = \frac{dx}{\dot{\chi}p} = -dp \left(\dot{\chi}x + \frac{\partial V}{\partial x} + \lambda\sqrt{\beta}\frac{\partial U}{\partial x} \right)^{-1} = \frac{df}{0}. \quad (\text{A.1})$$

Let us also assume that the distribution function $f(x, p; \theta_0)$ at some initial time θ_0 is given by

$$f(x, p; \theta_0) = f_0(x, p) = \mathcal{C} \left[\mathcal{H}(p - p_{(-)}^{(0)}(x)) - \mathcal{H}(p - p_{(+)}^{(0)}(x)) \right]. \quad (\text{A.2})$$

Suppose now that we have been able to solve the equations for the characteristics (A.1), and we have expressed the solution subsequently according to

$$p = P(x; \theta), \quad (\text{A.3})$$

where $P(x; \theta)$ is an appropriate function of its arguments. At each instant of time θ equation (A.3) defines two curves $p_{(+)}(x; \theta)$ and $p_{(-)}(x; \theta)$ in the phase space (x, p) , such that

$$p_{(+)}(x; \theta_0) = p_{(+)}^{(0)}(x), \quad p_{(-)}(x; \theta_0) = p_{(-)}^{(0)}(x). \quad (\text{A.4})$$

Therefore, if initially the distribution function is given by equation (A.2), then the solution of the Vlasov equation (2.12) at every subsequent instant of time θ is represented by the expression (2.14).

It is interesting to note that the chain of equations (excluding the last equation) for the characteristics (A.1) of the Vlasov equation formally coincide with those for the equation

$$\frac{\partial P}{\partial \theta} + \dot{\chi}P \frac{\partial P}{\partial x} = -\dot{\chi}x - \frac{\partial V}{\partial x} - \lambda\sqrt{\beta}\frac{\partial U}{\partial x}. \quad (\text{A.5})$$

As a matter of fact, equation (A.5) is equivalent to the two equations

$$\frac{\partial p_{(\pm)}}{\partial \theta} + \dot{\chi}p_{(\pm)} \frac{\partial p_{(\pm)}}{\partial x} = -\dot{\chi}x - \frac{\partial V}{\partial x} - \lambda\sqrt{\beta}\frac{\partial U}{\partial x}, \quad (\text{A.6})$$

with the initial conditions (A.4), provided U is defined as a solution to the equation (2.18).

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